Subtracting equation (11) from equation (10) we obtain

$$\frac{\partial j}{\partial z} = \frac{\rho_f - \rho_g}{\rho_f \rho_g} \cdot \frac{S}{E_g - E_f} - \frac{\alpha}{\rho_g} \cdot \frac{D_g \rho_g}{Dt} - \frac{1 - \alpha}{\rho_f} \cdot \frac{D_f \rho_f}{Dt}, \tag{12}$$

where, as in [1],

$$j = (1 - \alpha) v_f + \alpha v_g. \tag{13}$$

Returning to equation (10), the term $\partial/\partial z(\alpha v_g)$ can be written

$$\frac{\partial}{\partial z}[\alpha v_g] = \alpha \frac{\partial}{\partial z}[j + V_{gj}] + (j + V_{gj}) \frac{\partial \alpha}{\partial z}, \tag{14}$$

where, as in [1],

$$V_{gj} \equiv v_g - j. \tag{15}$$

With V_{gj} dependent only on α, as in [1],

$$\frac{\partial}{\partial z}[\alpha v_g] = \alpha \frac{\partial j}{\partial z} + \left(j + V_{gj} + \alpha \frac{\mathrm{d} V_{gj}}{\mathrm{d}\alpha}\right) \frac{\partial \alpha}{\partial z}. \tag{16}$$

Eliminating j and $\partial j/\partial z$ from equation (16) using equation (12) and substituting the result for $\partial/\partial z(\alpha v_g)$ into equation (10) we obtain

$$\begin{aligned} &\frac{\partial \alpha}{\partial t} + \left\{ v_{fi} + V_{gj} + \alpha \frac{\mathrm{d} V_{gj}}{\mathrm{d}\alpha} + \int_0^z \left[\frac{\rho_f - \rho_g}{\rho_f \rho_g} \cdot \frac{S}{E_g - E_f} \right.\right. \\ &\quad \left.\left. - \frac{1 - \alpha}{\rho_f} \cdot \frac{D_f \rho_f}{Dt} - \frac{\alpha}{\rho_g} \cdot \frac{D_g \rho_g}{Dt} \right] \mathrm{d}z \right\} \frac{\partial \alpha}{\partial z} \\ &\quad = \frac{\rho_m}{\rho_f \rho_g} \cdot \frac{S}{E_g - E_f} + \alpha(1 - \alpha) \left[\frac{1}{\rho_f} \cdot \frac{D_f \rho_f}{Dt} - \frac{1}{\rho_g} \cdot \frac{D_g \rho_g}{Dt} \right]. \end{aligned} \tag{17}$$

Equation (17), the void propagation equation, is identical to equation (21) of [1]. The quantity $S/(E_g - E_f)$ can be seen, by inspection of equation (7), to be identical to Γ_g, the vapour source term given by equation (22) of [1].

We see that the results of [1] can be derived entirely from a single continuity equation for the mixture together with the conventional energy conservation equation. Two continuity equations, one for each phase, are not necessary. The additional equation is brought into the analysis of [1, 2] by the introduction of an additional variable, that is, Γ_g, the vapour source term. Identifying $S/(E_g - E_f)$ as the vapour source term we see that equations (8) and (9) are the continuity equations for the vapour and liquid phases, respectively. Thus the two continuity equations are implied in the mixture continuity and the conventional energy conservation equations.

Zuber and Staub [2] say that basic differences exist between their void propagation equation and the void propagation equation of Kanai *et al.* [3] who start, as in this note, from the mixture continuity and the conventional energy conservation equations. Kanai *et al.* consider two different assumptions for the relative velocity between vapour and liquid, firstly that slip ratio is dependent only on the void fraction and secondly that slip velocity is dependent only on the void fraction. For the first assumption, the differences observed by Zuber and Staub can be traced to a comparison of incompatible equations. Equation (59) of [2], an equation in the average void across the duct, should have been used instead of equation (30) of [2], an equation in the local void. For the second assumption, the differences can be traced to an erroneous equation (9) of [3].

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N. Spinks

Australian Atomic Energy Commission
Research Establishment
Lucas Heights, N.S.W.
Australia

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REJOINDER

In the introduction to his letter [1], N. Spinks states:

(1) Zuber and Staub [2] place much emphasis on using two equations of continuity for analysis of boiling two-phase flow.

(2) Consequently it is most important that the question be resolved of what constitutes a sufficient number of conservation equations.

(3) Zuber and Staub derive a void propagation equation from three conservation equations, that is, continuity of liquid, continuity of vapour and a particular form of the conservation energy equation.

Following the derivation of his equation (17), Spinks concludes:

(4) We see that the results (of Zuber and Staub) can be derived entirely from a single continuity equation for the mixture together with the conventional energy conservation equation. Two continuity equations, one for each phase, are not necessary.

This statement and conclusion are misleading and in error because:

(1) The void propagation equation, i.e. equation (17) in [2] was not derived using the energy equation of the mixture.
Its derivation was based however on the continuity equation only, following the well established procedure used in deriving the diffusion equation.

(2) The void propagation equation is not a dependent equation, derived from the continuity equation of the mixture and the energy equation of the mixture.
It is however, an independent equation, based on the continuity equation of the vapor and analogous to the diffusion equation, which should be added to the three conservation equations (continuity, momentum and energy) of the mixture in order to properly describe a two-phase flow system.

(3) The vapor source term Γ_g in the void propagation equation, i.e. equation (17) of [2] cannot be evaluated from the energy equation of the mixture (except in a particular case discussed below).
This term, which is analogous to the source term in the continuity equation for a given species in a chemically reacting system, is specified however by a constitutive equation of evaporation appropriate to a particular flow regime.

In what follows we shall confirm the validity of these three comments.

1. THE VOID DIFFUSION AND VOID PROPAGATION EQUATIONS

It is indeed surprising that one would even question the need to use two continuity equations in order to formulate properly a two-phase flow system. Junior course text books on transport phenomena show (see for example, p. 560 in [3]) that multi-component systems, the number of continuity equations is equal to the number of components. It is customary to add these n continuity equations in a continuity equation of the mixture and to express the remaining $n-1$ as diffusion equations. But with one exception noted in [2]; this however was never done in analyses of boiling two-phase flow systems.

It is precisely because the great majority of analyses presently available are incomplete (as we have noted in [2]) that we introduced in the formulation the second continuity equation, i.e. that of the vapor. However, instead of expressing it as a void diffusion equation, we used the concept of kinematic waves to transform it into a void propagation equation [4] along the lines of [5]. This was done because data on diffusion coefficient in boiling mixtures are almost non-existent, whereas a method for determining the vapor drift velocity is available [6, 7]. It is evident from Spinks' comments that he neither appreciates the relation between the void propagation equation and the void diffusion equation (which was outlined in Appendix A of [2]), nor the significance and the method of deriving the void diffusion equation. It is desirable therefore to show the relation and derivation in more detail.

Using the nomenclature of [2], the continuity equations for the liquid and the vapor are given respectively by

$$\frac{\partial[(1-\alpha)\rho_f]}{\partial t}+\frac{\partial[(1-\alpha)\rho_f v_f]}{\partial z}=\Gamma_f \tag{1}$$

$$\frac{\partial(\alpha\rho_g)}{\partial t}+\frac{\delta(\alpha\rho_g v_g)}{\partial z}=\Gamma_g \tag{2}$$

where Γ_f and Γ_g are the rates of mass formation of the liquid and of the vapor per unit volume. The conservation of mass requires that

$$\Gamma_f+\Gamma_g=0. \tag{3}$$

The significance of the vapor source term Γ_g, given by a constitutive equation of net vaporization [7, 2] is discussed further in Section 3 below.

Carrying out the differentiation and eliminating $\partial\alpha/\partial t$ term between equation (1) and equation (2) and in view of equation (3) we obtain, for the one-dimensional problem

$$\frac{\partial}{\partial z}[(1-\alpha)v_f+\alpha v_g]=\frac{\partial j}{\partial z}=\frac{\Gamma_g}{\rho_g}\frac{(\rho_f-\rho_g)}{\rho_f}-\frac{(1-\alpha)}{\rho_f}\frac{D_f\rho_f}{Dt}-\frac{\alpha}{\rho_g}\frac{D_g\rho_g}{Dt}. \tag{4}$$

Consequently, the divergence of the volumetric flux density j, of the mixture is proportional to the rate of vapor mass formation per unit volume of the system and to the compressibilities of the liquid and of the vapor.

The derivation of either the void diffusion or of the void propagation equation follows the standard procedure (see for example, p. 518 in [8]).

We start with the continuity equation for the vapor and express the vapor velocity in terms of the velocity of the center of volume j, and of the vapor drift velocity V_{gj}, thus from equation (2), after differentiation

$$\alpha\left[\frac{\partial\rho_g}{\partial t}+v_g\frac{\partial\rho_g}{\partial z}\right]+\rho_g\left\{\frac{\partial\alpha}{\partial t}+\frac{\partial}{\partial z}[\alpha(j+V_{gj})]\right\}=\Gamma_g. \tag{5}$$

Differentiating the term which contains j, and in view of equation (4), we obtain after some rearrangement:

$$\frac{\partial\alpha}{\partial t}+j\frac{\partial\alpha}{\partial z}+\frac{\partial}{\partial z}(\alpha V_{gj})=\frac{\rho_m}{\rho_f}\frac{\Gamma_g}{\rho_g}+\alpha(1-\alpha)\left(\frac{1}{\rho_f}\frac{D_f\rho_f}{Dt}-\frac{1}{\rho_g}\frac{D_g\rho_g}{Dt}\right). \tag{6}$$

The third term on the left-hand side of this equation can be expressed in terms of the drift velocities of the vapor V_{gj} and of the liquid V_{fj}, thus

$$\frac{\partial(\alpha V_{gj})}{\partial z} = \frac{\partial}{\partial z}[\alpha(1-\alpha)(V_{gj} - V_{fj})]. \tag{7}$$

It is conventional procedure to express the right-hand side of this equation in terms of the diffusion coefficient D, thus

$$\frac{\partial}{\partial z}[\alpha(1-\alpha)(V_{gj} - V_{fj})] = -D\frac{\partial \alpha}{\partial z}. \tag{8}$$

Inserting equation (8) in equation (6) we obtain the void diffusion equation,

$$\frac{\partial \alpha}{\partial t} + j\frac{\partial \alpha}{\partial z} = \frac{\partial}{\partial z}\left(D\frac{\partial \alpha}{\partial z}\right) + \frac{\Gamma_g \rho_m}{\rho_g \rho_f} + \alpha(1-\alpha)\left(\frac{1}{\rho_f}\frac{D_f \rho_f}{Dt} - \frac{1}{\rho_g}\frac{D_g \rho_g}{Dt}\right). \tag{9}$$

If we neglect the vapor source term Γ_g and the two compressibilities we obtain equation (A-3) in [2], which is identical to the diffusion equation, i.e. to equation (8.1–11) in [8].

It is evident that in order to use the void diffusion equation, it is necessary to know the diffusion coefficient D. It was noted in [4, 5] that experimental data on the diffusion coefficient for boiling two-phase mixtures are almost non-existent. Furthermore, another difficulty may arise when the diffusion coefficient depends on the concentration α; in such a case equation (9) becomes a non-linear partial differential equation for which solutions are not usually available. It becomes necessary, therefore, to seek an alternate more amenable approach.

An alternate method was proposed in [5, 9] where the concept of kinematic waves was used instead of the diffusion coefficient. The application of kinematic waves to the present problem is advantageous because the analysis and results of [6, 7] provide a method for determining the vapor drift velocity V_{gj}. It is shown in [5, 6, 7] that for most flow regimes of practical interest the vapor drift velocity V_{gj} is either constant or a function of the void fraction α. Consequently we can express the third term on the left-hand side of equation (6) as

$$\frac{\partial(\alpha V_{gj})}{\partial z} = \left(V_{gj} + \alpha\frac{\partial V_{gj}}{\partial \alpha}\right)\frac{\partial \alpha}{\partial z}. \tag{10}$$

By substituting equation (10) in equation (6) we obtain the void propagation equation for a two-phase flow system with a change of phase [2, 4] thus

$$\frac{\partial \alpha}{\partial t} + C_K\frac{\partial \alpha}{\partial z} = \Omega \tag{11}$$

where the velocity of kinematic waves is

$$C_K = j + V_{gj} + \alpha\frac{\partial V_{gj}}{\partial \alpha} \tag{12}$$

and where the characteristic reaction frequency Ω is given by

$$\Omega = \frac{\rho_m \Gamma_g}{\rho_f \rho_g} + \alpha(1-\alpha)\left(\frac{1}{\rho_f}\frac{D_f \rho_f}{Dt} - \frac{1}{\rho_g}\frac{D_g \rho_g}{Dt}\right) \tag{13}$$

Equations (11–13) above are identical with equations (17, 19, 20) in [2]. It is seen that contrary to Spinks' statement, the void propagation equation is derived independently of the energy equation of the mixture.

2. TRADITIONAL FORMULATIONS

Spinks concludes that the void propagation equation can be derived from the continuity equation of the mixture and the energy equation of the mixture. He notes, furthermore, that two continuity equations are not necessary. These conclusions are not only erroneous but show also a lack of appreciation of the role of the diffusion equation in the formulation and analyses of multicomponent mixtures.

It was shown above that the derivation of the void propagation does not depend on the energy equation of the mixture. It is obtained however, from the continuity equation of the vapor following the well established procedure used in deriving the diffusion equation. It was noted also that for a system consisting of n components, the governing set of field equations consists of $n - 1$ diffusion equations in addition to the three conservation equations (continuity, momentum and energy) for the mixture. For a two-phase system, therefore, either the void diffusion or the void propagation equation is to be added to the latter three equations. Furthermore, we also know from elementary algebra, that a system is mathematically defined only if the number of variables is equal to the number of independent equations.

Let us examine now Spinks' conclusion. Were his conclusions correct, then obviously neither the void propagation equation, nor for that matter the void diffusion equation, would be an independent equation since, according to him, it can be derived from the continuity equation of the mixture and the energy equation of the mixture. Clearly were this the case, the governing set of independent equations would consist only of the three conservation equations for the mixture. Furthermore, from our understanding of multi-component mixtures we could anticipate that such system would be incomplete. This indeed is the case as discussed in [10].

It is shown in [10] that the traditional formulations based on the slip flow model which have been used heretofore to analyze dynamic characteristics of boiling systems are: (a) Incomplete and inadequate when applied to two-phase mixtures in thermal non-equilibrium, and (b) Inadequate for analyzing properly propagation phenomena and dynamic characteristics of boiling two-phase flow systems. The reasons are discussed in more detail in [10]. The set of field equations and of constitutive equations that is required to properly formulate the problem is considered further in [11, 12].

It can be concluded that were Spinks's conclusion correct, then neither the void propagation equation nor the void diffusion equation would be an independent equation. Were this the case, the governing set of equations would be incomplete.

3. CONSTITUTIVE EQUATION OF EVAPORATION

The error in Spinks' analysis and conclusions stems from his apparent belief that the vapor source term Γ_g, in the void propagation equation is identical to the term $S/E_g - E_f$, which he derives from the energy equation of the mixture and defines by equation (7) of [1].

We emphasized in [2] and the continuity equation for the vapor, i.e. equation (2) clearly indicates, that the vapor source term Γ_g has the same meaning as the mass source term in the continuity equation of a given species undergoing a chemical reaction. It is well known that in chemically reacting mixtures the source term is specified by an appropriate constitutive equation for chemical reaction. We noted therefore in [2] that in order to specify Γ_g in boiling mixtures, it is necessary to specify the appropriate constitutive equation of evaporation.

It was also discussed in [2] that such a constitutive equation of evaporation will depend on the mode of heat transfer in the vicinity of the vapor–liquid interface and on the geometry of this interface. Consequently it will change with a change of flow regimes.

For example, in nucleate boiling the constitutive equation of evaporation, i.e. Γ_g, depends on (1) the rate of bubble nucleation, (2) the rate of bubble growth and (3) the bubble population density [7, 10]. Whereas in the fog flow regime Γ_g, depends on the droplet number density and on the rate of droplet evaporation [10]. A constitutive equation of evaporation appropriate to adiabatic flashing flows was first derived in [13], is considered further in [14]. Whereas a constitutive equation of condensation appropriate to adiabatic condensing nozzle flow was first formulated in [15].

Spinks is mistaken if he believes that the vapor source term Γ_g is generally specified by the energy equation for the mixture, i.e. equation (12) in [2]. We were careful to note in [2] that for the purpose of that paper only Γ_g could be determined from equation (12) of [2]. This was possible because in [2] we considered a mixture in thermal equilibrium. It is only for such a mixture that the energy equation of the mixture can be used to evaluate Γ_g and yield equation (41) which was used throughout the analysis of [2].

Clearly in general, the vapor source term Γ_g cannot be determined from the energy equation of the mixture but must be evaluated from a constitutive equation of evaporation appropriate to a particular flow regime. Indeed it is shown in [10] that such an equation for Γ_g, determines the level of thermal non-equilibrium of a two-phase flow system.

It is evident that the term $S/E_g - E_f$, defined by equation (7) of [1] and obtained therefore from the energy equation of the mixture, does not correspond to the constitutive equation of evaporation. Consequently Spinks' conclusion that the void propagation equation can be derived from the energy equation of the mixture is erroneous.

4. CONCLUSION

In view of the foregoing it can be concluded that Spinks' statements and conclusions are misleading and erroneous.

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NOVAK ZUBER

Department of Mechanical Engineering,
New York University
University Heights
New York, New York 10453
U.S.A.